Exercise 7.2.13

Radioactive nuclei decay according to the law

$$\frac{dN}{dt} = -\lambda N,$$

N being the concentration of a given nuclide and λ, the particular decay constant. In a radioactive series of two different nuclides, with concentrations $N_1(t)$ and $N_2(t)$, we have

$$\begin{aligned}\frac{dN_1}{dt} &= -\lambda_1 N_1, \\ \frac{dN_2}{dt} &= \lambda_1 N_1 - \lambda_2 N_2.\end{aligned}$$

Find $N_2(t)$ for the conditions $N_1(0) = N_0$ and $N_2(0) = 0$.

Solution

Begin by solving the ODE for $N_1(t)$. Divide both sides by N_1.

$$\frac{\frac{dN_1}{dt}}{N_1} = -\lambda_1$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dt}(\ln N_1) = -\lambda_1$$

Integrate both sides with respect to t.

$$\ln N_1 = -\lambda_1 t + C_1$$

Exponentiate both sides.

$$\begin{aligned}N_1(t) &= e^{-\lambda_1 t + C_1} \\ &= e^{-\lambda_1 t} e^{C_1}\end{aligned}$$

Apply the initial condition $N_1(0) = N_0$ to determine e^{C_1}.

$$N(0) = e^{-\lambda_1(0)} e^{C_1} \quad \rightarrow \quad N_0 = e^{C_1}$$

So then

$$N_1(t) = N_0 e^{-\lambda_1 t}.$$

Substitute this formula for N_1 into the second ODE involving N_2.

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2$$

$$\frac{dN_2}{dt} = \lambda_1 N_0 e^{-\lambda_1 t} - \lambda_2 N_2$$

Bring $\lambda_2 N_2$ to the left side.

$$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_0 e^{-\lambda_1 t}$$

This is a first-order linear ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \lambda_2\,ds\right) = e^{\lambda_2 t}$$

Proceed with the multiplication.

$$e^{\lambda_2 t}\frac{dN_2}{dt} + \lambda_2 e^{\lambda_2 t}N_2 = \lambda_1 e^{\lambda_2 t}N_0 e^{-\lambda_1 t}$$

The left side can now be written as a derivative by the product rule. Combine the exponential functions on the right side.

$$\frac{d}{dt}(e^{\lambda_2 t}N_2) = N_0\lambda_1 e^{(\lambda_2-\lambda_1)t} \tag{1}$$

Suppose first that $\lambda_1 \neq \lambda_2$. Integrate both sides with respect to t.

$$e^{\lambda_2 t}N_2 = N_0\frac{\lambda_1}{\lambda_2-\lambda_1}e^{(\lambda_2-\lambda_1)t} + C_2$$

Divide both sides by $e^{\lambda_2 t}$.

$$\begin{aligned} N_2(t) &= N_0\frac{\lambda_1}{\lambda_2-\lambda_1}e^{(\lambda_2-\lambda_1)t}e^{-\lambda_2 t} + C_2 e^{-\lambda_2 t} \\ &= N_0\frac{\lambda_1}{\lambda_2-\lambda_1}e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t} \end{aligned}$$

Apply the initial condition $N_2(0) = 0$ now to determine C_2.

$$N_2(0) = N_0\frac{\lambda_1}{\lambda_2-\lambda_1}e^{-\lambda_1(0)} + C_2 e^{-\lambda_2(0)}$$

$$0 = N_0\frac{\lambda_1}{\lambda_2-\lambda_1} + C_2$$

$$C_2 = -N_0\frac{\lambda_1}{\lambda_2-\lambda_1}$$

Plug this back into the general solution for $N_2(t)$.

$$\begin{aligned} N_2(t) &= N_0\frac{\lambda_1}{\lambda_2-\lambda_1}e^{-\lambda_1 t} - N_0\frac{\lambda_1}{\lambda_2-\lambda_1}e^{-\lambda_2 t} \\ &= N_0\frac{\lambda_1}{\lambda_2-\lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}), \qquad \lambda_1 \neq \lambda_2 \end{aligned}$$

Suppose secondly that $\lambda_1 = \lambda_2$. Then equation (1) becomes

$$\frac{d}{dt}(e^{\lambda_2 t}N_2) = N_0\lambda_2.$$

Integrate both sides with respect to t.

$$e^{\lambda_2 t}N_2 = N_0\lambda_2 t + C_3$$

Divide both sides by $e^{\lambda_2 t}$.

$$N_2(t) = e^{-\lambda_2 t}(N_0\lambda_2 t + C_3)$$

Apply the initial condition $N_2(0) = 0$ to determine C_3.

$$N_2(0) = C_3 = 0$$

So then

$$N_2(t) = N_0\lambda_2 t e^{-\lambda_2 t}, \qquad \lambda_1 = \lambda_2.$$

Therefore,

$$\begin{aligned} N_1(t) &= N_0 e^{-\lambda_1 t} \\ N_2(t) &= \begin{cases} N_0 \dfrac{\lambda_1}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}) & \text{if } \lambda_1 \neq \lambda_2 \\ \\ N_0 \lambda_2 t e^{-\lambda_2 t} & \text{if } \lambda_1 = \lambda_2 \end{cases}. \end{aligned}$$